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LETTER TO THE EDITOR

Randomly stirred fluids, mode coupling theories and the turbulent Prandtl number

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Abstract. The model of a randomly stirred fluid introduced by De Dominicis and Martin is used to study the convection of a passive scalar, namely temperature. Mode coupling theory of dynamic critical phenomena is used to calculate the turbulent Prandtl number, which is a universal amplitude ratio. The result agrees with that obtained from the dynamic renormalisation group.

A model for studying the long-wavelength properties of a randomly stirred fluid was put forward by Forster *et al* (1977) and generalised by De Dominicis and Martin (1979). These authors calculated the effect of the non-linear terms in the Navier–Stokes equation on the bare viscosity and showed that for a certain class of forcing, the long-wavelength viscosity diverged and for a particular forcing (maximally random in a certain sense) the effective viscosity was such that the Kolmogorov spectrum for the energy density of a turbulent fluid was obtained. Recently Yakhot and Orszag (1986) have calculated the associated amplitude ratio (viscosity coefficient to the strength of the forcing) and using some existing results (e.g. Leslie 1972) as an extra ingredient evaluated the Kolmogorov constant and found it in remarkable agreement with experiment. They then applied the dynamical renormalisation group methods of Ma and Mazenko (1975) to the coupled problem of randomly forced fluid and temperature diffusion and found a turbulent Prandtl number (one of the phenomenological milestones akin to the Kolmogorov law) quite close to the accepted experimental value. In this letter, we produce a derivation of the turbulent Prandtl number using the Kubo formula for diffusion and mode coupling theories of critical dynamics (Kawasaki 1970, Ferrell 1970) and show that it is in exact agreement with the value obtained from the dynamic renormalisation group calculation of Yakhot and Orszag. This, in itself, is not surprising. However, the mode coupling calculation presented here makes the associated approximation transparent, and also shows how the direct-interaction approximation integrals (Leslie 1972) need to be handled to reproduce the results of the renormalisation group. It thus becomes clear that one is talking about a long-wavelength property and the near perfect agreement with the experimental result should be viewed with a healthy suspicion.

The model proposed by De Dominicis and Martin and extended to allow for thermal diffusion is

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\frac{\nabla P}{\rho}+\nu\tilde{\nabla}\boldsymbol{v}+\boldsymbol{f} \tag{1}$$

$$\nabla\cdot\boldsymbol{v}=0 \tag{2}$$

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$$\frac{\partial T}{\partial t}+(\boldsymbol{v}\cdot\nabla)T=D\nabla^2 T. \tag{3}$$

In the above $\boldsymbol{v}$ and T are the velocity and temperature fields, ν and D are the kinematic viscosity and thermal diffusivity respectively and P and ρ are the pressure and density (constant because of incompressibility). The pressure can be eliminated from (1) by using (2). The random stirring force is specified by its correlation; in momentum and frequency space this correlation is

$$\langle f_i(\boldsymbol{k},\omega)f_j(\boldsymbol{k}',\omega')\rangle=2D_0(2\pi)^{d+1}P_{ij}(k)k^{4-D}(1+k^2L^2)^{-y/2}\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(\omega+\omega') \tag{4}$$

$$P_{ij}(k)=\delta_{ij}-k_ik_j/k^2 \tag{5}$$

L is a stirring length, and d is the dimensionality of space.

The perturbation theory in diagrammatic terms for (1) was established by Wyld (1961). The dressed single loop diagram for the response function leads to the effective zero-frequency viscosity or self-energy $\Sigma(k)$ given by

$$\Sigma(k)=\frac{4D_0}{d-1}\int\frac{\mathrm{d}^dp\,\mathrm{d}\omega}{(2\pi)^{d+1}}\frac{1}{p^{d-4}(1+p^2L^2)^{y/2}}f(p,\boldsymbol{q},\boldsymbol{k})\left|\frac{1}{-\mathrm{i}\omega+p^2\Sigma(p)}\right|^2\frac{1}{\mathrm{i}\omega+q^2\Sigma(q)} \tag{6}$$

where

$$\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q} \tag{7}$$

and

$$f(p,\boldsymbol{q},\boldsymbol{k})=\sum_{i,j,m,r,s}P_{ijm}(k)P_{jr}(p)P_{ms}(q)P_{sri}(q) \tag{8}$$

with

$$P_{abc}(k)=k_bP_{ac}(k)+k_cP_{ab}(k) \tag{9}$$

and the summation over i, j, m, r, s in (8) runs from 1 to d. The summation in (8) can be performed to yield

$$f(p,q,k)=\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^3}{k^4q^2}-\frac{(\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{k}\cdot\boldsymbol{p})}{p^2k^2}+\left(\frac{d-3}{2}\right)\left(1-\frac{(\boldsymbol{p}\cdot\boldsymbol{k})^2}{p^2k^2}\right). \tag{10}$$

Performing the frequency convolution in (7) leads to

$$\Sigma_{(k)}=\frac{2\tilde{D}_0}{d-1}\int\frac{\mathrm{d}^dp}{(2\pi)^d}\frac{f(p,q,k)}{p^{d-2+y}}\frac{1}{\Sigma(p)(p^2\Sigma(p)+q^2\Sigma(q))} \tag{11}$$

where we have assumed that the relevant region of momentum space is such that $pL\gg 1$ (distance scales much smaller than L) and have absorbed the factor of L^y in D_0 to define $\tilde{D}_0L^{-y}$. We now note that a self-consistent solution of (11) is obtained for

$$\Sigma(k)=\Gamma_0k^x \tag{12}$$

where

$$x=-y/3 \tag{13}$$

and

$$\Gamma_0^3=\frac{2\tilde{D}_0}{d-1}\int\frac{\mathrm{d}p'}{(2\pi)^d}\frac{f(p,q,k)}{(p')^{d+y/3}}\left(1+\left|\frac{q'}{p'}\right|^{2-y/3}\right)^{-1} \tag{14}$$

with all the primed momenta scaled by k.

The Kolmogorov case is obtained for $y=4$ (for a different discussion see Ronis (1977)).

We now point out that the integral in (14) would give a singularity in Γ_0^3 for $y\geqslant 3$, the singularity coming from the region of momentum space where $p\ll k$ and this is the difficulty that has plagued the DIA.

However, the integral of (14) has an infrared divergence for $y\geqslant 0$, the divergence coming from the region of momentum space $p\gg k$. To obtain the long-wavelength behaviour of the viscosity, however, we need to extract the infrared divergence and (14) can be written as

$$\Gamma_0^3=\tilde{D}_0[I_0/y+I_1+yI_2+\mathrm{O}(y^2)] \tag{15}$$

where the integral I_0 is the integral in the limit of $p\gg k$ and the remaining numbers I_1, I_2, etc, are obtained by a systematic expansion of the integral about $y=0$. We claim that this procedure establishes a systematic expansion for the quantity $\Gamma_0^3/\tilde{D}_0$ in powers of y when all loops are taken into account. The two-loop graph has the structure (contribution to Γ_0) D_0^2V/Γ_0^3, where V is the two-loop integral which from a power counting argument is seen to have an infrared divergence at $y=0$. Thus V has the expansion $V_0/y+V_1+\mathrm{O}(y)$ and noting that D_0/Γ_0^3 is $\mathrm{O}(1/y)$ from (15), we conclude that the leading contribution of the two-loop graph is at O(1). The leading-order term from three-loop graphs can easily be seen to contribute at $\mathrm{O}(y)$ and thus the loop expansion is an expansion in y. Thus, to lowest order in y the amplitude ratio $\Gamma_0^3/\tilde{D}_0$ is given by

$$\Gamma_0^3/\tilde{D}_0=I_0/y \tag{16}$$

where explicit evaluation yields

$$I_0=\frac{3}{2}\frac{C_d}{(2\pi)^d}\frac{d-1}{d+2} \tag{17}$$

C_d being the surface area of a d-dimensional sphere.

We now turn to (3). The diffusion coefficient or the dressed single-loop self-energy can be obtained from a perturbation theory for the temperature response function or the Kubo formulae for transport coefficients. The result, well known in critical dynamics of single component fluids (e.g. Kawasaki 1970), is

$$\begin{aligned} D(k)&=2\tilde{D}_0\int\frac{\mathrm{d}^dp}{(2\pi)^d}\frac{\mathrm{d}\omega}{2\pi}\frac{\sin^2\theta}{p^{d-4+y}}\left|\frac{1}{-\mathrm{i}\omega+p^2\Sigma(p)}\right|^2\frac{1}{\mathrm{i}\omega+q^2D(q)}\\ &=\tilde{D}_0\int\frac{\mathrm{d}^dp}{(2\pi)^d}\frac{\sin^2\theta}{p^{d-2+y}}\frac{1}{(p^2\Sigma(p)+q^2D(q))}. \end{aligned} \tag{18}$$

We see immediately that if $\Sigma(p)$ scales as $p^{-y/3}$, $D(q)$ must scale as $q^{-y/3}$ (extended dynamic scaling; Ferrell *et al* (1968)). Hence, with

$$D(k)=\Gamma_1k^{-y/3} \tag{19}$$

we have

$$\Gamma_1=\frac{\tilde{D}_0}{\Gamma_0^2}\int\frac{\mathrm{d}^dp}{(2\pi)^d}\frac{\sin^2\theta}{p^{d+y/3}}\left(1+\frac{\Gamma_1}{\Gamma_0}\left|\frac{q}{p}\right|^{2-y/3}\right)^{-1}. \tag{20}$$

Once again, we notice that the integral has an infrared divergence for $y=0$ and can be expanded as in (15). A loop expansion can be carried out for Γ_1 exactly as for Γ_0

and, to the leading order in y, we find

$$\Gamma_1\Gamma_0^2\left(1+\frac{\Gamma_1}{\Gamma_0}\right)=\tilde{D}_0\frac{3}{y}\frac{C_D}{(2\pi)^D}\frac{d-1}{d}. \tag{21}$$

From (16), (17) and (21),

$$\frac{\Gamma_0}{\Gamma_1}\left(1+\frac{\Gamma_1}{\Gamma_0}\right)^{-1}=\frac{d}{2(d+2)}. \tag{22}$$

The turbulent Prandtl number σ_t is given by

$$\sigma_t=\frac{\Sigma(k)}{D(k)}=\frac{\Gamma_0}{\Gamma_1}. \tag{23}$$

For $d=3$, (22) yields

$$\sigma_t=0.72 \tag{24}$$

in accordance with the result obtained from the dynamic renormalisation group. The experimental value for σ_t lies between 0.7 and 0.9 and thus (24) is a remarkably good estimate. We note in passing that (17), used together with the result obtained from the energy balance (Leslie 1972), gives a Kolmogorov constant in exact accordance with Yakhot and Orszag (1986).

As expected, the conventional mode coupling treatment of the non-linear terms in the Navier–Stokes equation and heat diffusion equation yields exponents and amplitudes in agreement with the renormalisation group. The long-wavelength nature of the approximation is made explicit in the discussion following (14). This corresponds to the fact that the controlling fixed point in the dynamic renormalisation group is an infrared fixed point. One has to keep this fact in mind while discussing the relevance of the result to the actual situation in turbulence. The Kolmogorov law and the universal turbulent Prandtl number are valid in the inertial range—a range of wavevectors which lies intermediate between the range where energy is fed in ($\sim L^{-1}$) and where it is dissipated ($\sim \nu^{3/4}/\varepsilon^{1/4}$, ε being the rate of dissipation of energy). The results in this letter hold for wavenumbers which are small, but larger than L^{-1}; how far these wavenumbers can extend into the inertial range is a problem worth a closer look.

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